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Effect of intercommunication between various elements of a material on its crack extension resistance: linear intercommunication force law

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A material's resistance to failure by crack extension is influenced by the characteristics of the intercommunication between the various elements that comprise the material's structure. This paper investigates in detail the case where the material consists of linear elements with the crack being transverse to these elements. Various simulation models are analysed, with consideration being focused on the case where the shear intercommunication between the elements is linearly related to the shear strain. It is shown that there is a correlation between weak intercommunication and a high crack extension resistance but that, for a wide range of situations, the strength of the shear intercommunication has relatively little effect on the crack extension resistance. This means that when seeking explanations for large differences in the crack extension resistances of various types of material, it is appropriate to look for such explanations beyond the confines of a simple linear intercommunication law.

1. Introduction

Materials can often be regarded as consisting of a collection of elements between which there is some shear intercommunication; examples are crystalline metals, rope, cloth, natural tendons, and wood. Gordon [1] has emphasized that the characteristics of this intercommunication play a crucial role with regard to the material's resistance to crack extension, and has identified various categories of intercommunication in terms of their effects on crack extension. This paper examines the situation where the material is comprised of a series of linear elements for the case where the crack is transverse to the elements. Various simulation models are analysed, with consideration being focused on the situation where the shear intercommunication between the elements is linearly related to the shear strain. It is shown that there is a correlation between weak intercommunication and a high crack extension resistance but that, for a wide range of situations, the strength of the shear intercommunication has relatively little effect on the crack extension resistance. Consequently, when seeking explanations for large differences in the crack extension resistances of various types of material, it is appropriate to look for such explanations beyond the confines of a simple linear intercommunication law.

2. Theoretical analysis

The first very simple two-dimensional model simulating the effect of a linear intercommunication between the various elements of a material on the extension of a crack is illustrated in Fig. 1. There are three vertical linear elastic elements each of length $2h$, spacing b, and tensile modulus M. To simulate a displacement applied to the system, the ends of each element are subjected to applied displacements $\pm D_*$. The central element is assumed to have fractured at its mid-point, and the crack extension process is simulated by the fracturing of the outer elements at their mid-points, it being assumed that this requires the attainment of a critical fracture strain, ε_F. It must be emphasized that this model is extremely idealized in the way in which it simulates the crack extension process, because only three elements are involved in the cracking process. Nevertheless, a very simple analysis is possible and this allows for some key features of the problem to be readily appreciated. The model is an improvement of an earlier model [2, 3] where the central element is intact and the outer elements are fractured, with the crack extension process being simulated by the fracturing of the central element. The shear communication between the elements is simulated by a restraining force between adjacent elements that is linearly related to the relative displacement of equivalent points in the adjacent elements.

With x being measured from the mid-points of the elements, the displacements $u_1(x)$ of the outer elements and the displacement $u_0(x)$ of the central element are odd functions of x. If $T_1(x)$ is the tension within an outer element, $T_0(x)$ is the tension within the central element, and $L\phi/ab \equiv L(u_0 - u_1)/ab$ is the restraining force per unit length provided by the shear intercommunication between the elements (a is a materials-related length parameter associated with each element), equilibrium of the elements provides the equations

$$\frac{dT_1}{dx} + \frac{L(u_0 - u_1)}{ab} = 0 \qquad (1)$$

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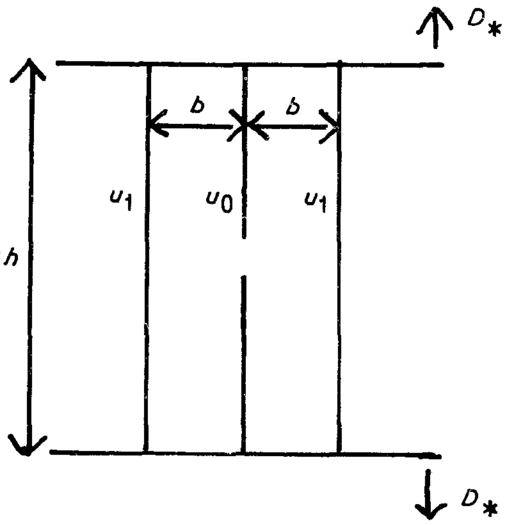

Figure 1 The simple model simulating the effect of the shear intercommunication between the various elements of a material on the extension of a crack. $2h$ is the length of each element and b is the spacing between the elements.

$$\frac{dT_0}{dx} - \frac{2L(u_0 - u_1)}{ab} = 0 \quad (2)$$

with

$$T_1 = M\frac{du_1}{dx} \quad (3)$$

and

$$T_0 = M\frac{du_0}{dx} \quad (4)$$

whereupon Equations 1 and 2 become, respectively,

$$\frac{d^2u_1}{dx^2} + \frac{L(u_0 - u_1)}{Mab} = 0 \quad (5)$$

$$\frac{d^2u_0}{dx^2} - \frac{2L(u_0 - u_1)}{Mab} = 0 \quad (6)$$

By virtue of the system's symmetry it is necessary to consider only one half of the system (i.e. $x > 0$) for which the boundary conditions are

$$u_0, u_1 = D_* \quad \text{when } x = h \quad (7)$$

$$\frac{du_0}{dx}, u_1 = 0 \quad \text{when } x = 0 \quad (8)$$

the condition $du_0/dx = 0$ when $x = 0$ reflecting the fact that the central element is fractured at its mid-point.

It follows from Equations 5 and 6 that

$$u_0 = u_1 - \frac{Mab}{L}\frac{d^2u_1}{dx^2} \quad (9)$$

and

$$\frac{d^4u_1}{dx^4} - \frac{3L}{Mab}\frac{d^2u_1}{dx^2} = 0 \quad (10)$$

the solutions of these linear differential equations satisfying the boundary conditions Equations 7 and 8 being

$$u_0 = A + Bx - 2C\exp(-\lambda x) - 2D\exp(\lambda x) \quad (11)$$

$$u_1 = A + Bx + C\exp(-\lambda x) + D\exp(\lambda x) \quad (12)$$

where the constants A, B, C and D are given by the expressions

$$A = \frac{D_*[\exp(2\lambda h) - 1]}{[\exp(2\lambda h) - 1] + 2\lambda h[\exp(2\lambda h) + 1]} \quad (13a)$$

$$B = \frac{2\lambda D_*[\exp(2\lambda h) + 1]}{[\exp(2\lambda h) - 1] + 2\lambda h[\exp(2\lambda h) + 1]} \quad (13b)$$

$$C = \frac{-D_*\exp(2\lambda h)}{[\exp(2\lambda h) - 1] + 2\lambda h[\exp(2\lambda h) + 1]} \quad (13c)$$

$$D = \frac{D_*}{[\exp(2\lambda h) - 1] + 2\lambda h[\exp(2\lambda h) + 1]} \quad (13d)$$

and λ is equal to $(3L/Mab)^{1/2}$. If the $\varepsilon_M \equiv D_*/h$ is regarded as the macroscopic strain and $\varepsilon_L = (du_1/dx)_{x=0}$ is the local strain at the mid-point of an unfractured element (it is assumed that an element fractures when ε_L attains a critical value ε_F), Equations 12 and 13 show that

$$\frac{\varepsilon_L}{\varepsilon_M} = \frac{3}{\left[2 + \frac{\tanh(\lambda h)}{\lambda h}\right]} \quad (14)$$

It is immediately seen that the ratio $\varepsilon_L/\varepsilon_M$, which reflects the extent to which the strain is focused, decreases as the intercommunication (L or λ) becomes weaker, leading to an increased crack extension resistance. However, $\varepsilon_L/\varepsilon_M$ always lies between 1 and 3/2 and is close to 3/2 (the value appropriate to $h = \infty$) for a wide range of h values, i.e. it is relatively unaffected by the strength of the shear intercommunication.

Suppose, instead, that the boundary conditions at $x = h$ are $du_0/dx = \varepsilon_A$, which is equivalent to specifying that constant forces $P_A \equiv M\varepsilon_A$ are applied to the system. In this case, by following procedures similar to those in the preceding paragraph, the general Equations 11 and 12 together with the new boundary conditions give

$$\frac{\varepsilon_L}{\varepsilon_A} = \frac{3}{2} \quad (15)$$

a result that can also be obtained merely by balancing forces; in this case the magnitude of the strain-focusing effect is unaffected by the strength of the shear intercommunication. Comparison of Equations 14 and 15 show that for the limiting situation where the elements are infinitely long, i.e $h \to \infty$, the magnitude of the strain-focusing effect is independent of the strength of the shear intercommunication and, furthermore, is the same irrespective of whether a macroscopic or far-field strain is applied to the system. Similar conclusions were obtained [3] for the model in which the central element is intact and the two outer elements are fractured.

To demonstrate the generality of the conclusion

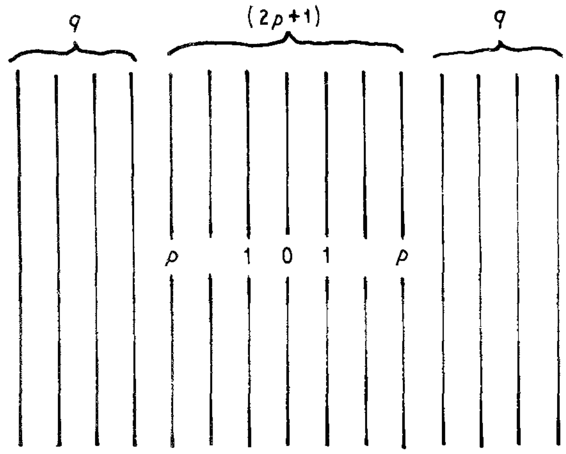

Figure 2 The model with $(2p + 1)$ fractured elements and $2q$ intact elements; the elements have infinite length.

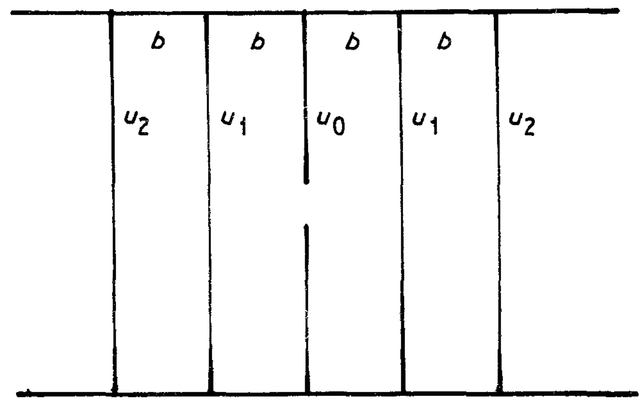

Figure 3 The model where there is one fractured element and four intact elements; the elements have infinite length.

regarding the independency of the strength of the shear intercommunication when the element length is infinite, consider the model (Fig. 2) where there are $(2p + 1)$ fractured elements and $2q$ intact elements. For this model, the governing force equilibrium equations that are analogous to Equations 5 and 6 are

$$\frac{d^2u_0}{dx^2} - \frac{2L(u_0 - u_1)}{Mab} = 0 \qquad (16)$$

$$\frac{d^2u_n}{dx^2} - \frac{L}{Mab}\{(u_{n-1} - u_n) - (u_n - u_{n+1})\} = 0 \qquad \begin{bmatrix} n = 1 \text{ to} \\ p + q - 1 \end{bmatrix} \qquad (17)$$

$$\frac{d^2u_{p+q}}{dx^2} + \frac{L}{Mab}\{(u_{p+q-1} - u_{p+q})\} = 0 \qquad (18)$$

where $u_n(x)$ is the displacement of the nth element, and x is measured from the mid-point of an element. The boundary conditions are $du_n/dx = 0$ for $n = 0, 1 \ldots, p$ and $u_n = 0$ for $n = p + 1, p + 2, \ldots, p + q$, when $x = 0$, and in accord with the earlier comments concerning the model in Fig. 1, the boundary condition $du_n/dx = \varepsilon_M$ when $x \to \infty$ implies a macroscopic strain ε_M, far-field strain ε_M or constant applied forces $P = M\varepsilon_M$. Introducing new dimensionless variables $w_n = u_n/b$ for all n and $y = x(L/Mab)^{1/2}$, the system of Equations 16, 17 and 18 reduces to a system of linear differential equations involving w and the variable y but not the parameters L and M. The boundary conditions become $dw_n/dy = 0$ for $n = 0, 1, \ldots, p$ and $w_n = 0$ for $n = p + 1, p + 2, \ldots, p + q$ when $y = 0$, and $dw_n/dy = \varepsilon_M(Ma/Lb)^{1/2}$ for all n when $y = \infty$. Simple dimensional reasoning then shows that the functional relationships for the displacements are

$$w_n = \varepsilon_M\left(\frac{Ma}{Lb}\right)^{1/2} f_n(y) \qquad (19)$$

or in terms of the original variables

$$\frac{u_n}{b} = \varepsilon_M\left(\frac{Ma}{Lb}\right)^{1/2} f_n\left[\frac{x}{\left(\frac{Mab}{L}\right)^{1/2}}\right] \qquad (20)$$

Because the strain at the mid-point of the first intact element, i.e. the element for which $n = p + 1$, is du_{p+1}/dx for $x = 0$, it follows from Equation 20 with $n = p + 1$ that this strain is independent of the ratio (L/M), i.e. independent of the strength of the shear intercommunication. This simple dimensional approach does not, of course, give the magnitude of the strain-focusing effect, but an analysis is fairly simple when there is one fractured element and four intact elements as the next paragraph now shows.

Consider the model (Fig. 3) where the elements are infinitely long, there being one central fractured element and four intact elements, two on either side of the fractured element. With this model, the governing force equilibrium conditions are

$$\frac{d^2u_0}{dx^2} + \frac{L}{Mab}[2u_1 - 2u_0] = 0 \qquad (21)$$

$$\frac{d^2u_1}{dx^2} + \frac{L}{Mab}[u_0 - 2u_1 - u_2] = 0 \qquad (22)$$

$$\frac{d^2u_2}{dx^2} + \frac{L}{Mab}[u_1 - u_2] = 0 \qquad (23)$$

where $u_n(x)$ is the displacement of the nth element. The boundary conditions are $du_0/dx = 0$ and u_1, $u_2 = 0$ when $x = 0$, while the condition $du_0/dx = du_1/dx = du_2/dx = \varepsilon_M$ when $x \to \infty$ implies a macroscopic strain ε_M, a far-field strain ε_M, or constant applied forces $P = M\varepsilon_M$. Again introducing the new dimensionless variables $w_n = u_n/b$ and $y = x(L/Mab)^{1/2}$, the system of Equations 21, 22 and 23 reduce to

$$\frac{d^2w_0}{dy^2} + [2w_1 - 2w_0] = 0 \qquad (24)$$

$$\frac{d^2w_1}{dy^2} + [w_0 - 2w_1 + w_2] = 0 \qquad (25)$$

$$\frac{d^2w_2}{dy^2} + [w_1 - w_2] = 0 \qquad (26)$$

coupled with the boundary conditions $dw_0/dy = 0$, $w_1 = w_2 = 0$ when $y = 0$, while $dw_0/dy = dw_1/dy = dw_2/dy = \varepsilon_M(Ma/Lb)^{1/2}$ when $y = \infty$. It follows from Equations 24, 25 and 26 that the displacements w_0, w_1 and w_2 can be expressed in the forms

$$w_0 = A + By + C\exp(-m_1 y) + D\exp(-m_2 y) \quad (27)$$

$$w_1 = A + By + C\left[1 - \frac{m_1^2}{2}\right]\exp(-m_1 y) + D\left[1 - \frac{m_2^2}{2}\right]\exp(-m_2 y) \quad (28)$$

$$w_2 = A + By + C\left[1 - 2m_1^2 + \frac{m_1^4}{2}\right] \times \exp(-m_1 y) + D\left[1 - 2m_2^2 + \frac{m_2^4}{2}\right] \times \exp(-m_2 y) \quad (29)$$

where A, B, C and D are constants that can be determined from the boundary conditions, while m_1 and m_2 are the positive solutions of the quartic equation

$$m^4 - 5m^2 + 5 = 0 \quad (30)$$

The boundary conditions give

$$B = \varepsilon_M (Ma/Lb)^{1/2} \quad (31)$$

$$C = \frac{Bm_2(3 - m_2^2)}{m_1(m_2 - m_1)(3 - m_1^2 - m_1 m_2 - m_2^2)} \quad (32)$$

$$D = \frac{Bm_1(3 - m_1^2)}{m_2(m_1 - m_2)(3 - m_2^2 - m_1 m_2 - m_1^2)} \quad (33)$$

It follows that the local strain at the centre of the first intact element is $\varepsilon_{L1} = \mathrm{d}u_1/\mathrm{d}x = (Lb/Ma)^{1/2}\mathrm{d}w_1/\mathrm{d}y$ for $y = 0$, and is given from Equations 28, 31, 32 and 33 by the expression

$$\varepsilon_{L1} = \varepsilon_M m_1 m_2 \Big/ 2\left[\frac{(m_1 + m_2)^2}{(m_1 m_2 + 3)} - 1\right] \quad (34)$$

or

$$\varepsilon_{L1} = \frac{\varepsilon_M(5 - 5^{1/2})}{2} = 1.38\varepsilon_M \quad (35)$$

using Equation 30. This result shows the extent to which the strain is focused as a consequence of the linear shear intercommunication. For the sake of comparison, the local strain at the centre of the second intact element is $\varepsilon_{L2} = \mathrm{d}u_2/\mathrm{d}x = (Lb/Ma)^{1/2}\mathrm{d}w_2/\mathrm{d}y$ for $y = 0$, and is given from Equations 29, 31, 32 and 33 by the expression

$$\varepsilon_{L2} = \left\{\varepsilon_M m_1 m_2 \Big/ 2\left[\frac{(m_1 + m_2)^2}{(m_1 m_2 + 3)} - 1\right]\right\} \times \left\{1 + (m_1 m_2 + 3) - \frac{3(m_1 + m_2)^2}{(m_1 m_2 + 3)}\right\} \quad (36)$$

or

$$\varepsilon_{L1} = \frac{\varepsilon_M 5^{1/2}}{2} = 1.12\varepsilon_M \quad (37)$$

Balance of forces gives $(\varepsilon_{L1} + \varepsilon_{L2}) = 5\varepsilon_M/2$, a result that is compatible with Equations 35 and 37.

The analyses for the preceding discrete element models have clearly shown that the strain-focusing effect, and therefore the material's crack extension resistance, is independent of the strength of the shear intercommunication for a linear intercommunication law, when the elements are infinitely long. This conclusion is also valid when the behaviour of the individual elements is averaged, so as to give a continuum-type model, as the following analysis now shows. Consider the case where the elements, whose behaviour is to be averaged, are paralled to the y-axis. The deformation is plane strain with the deformation confined to the xy plane and the only displacement component v is parallel to the y-axis (note that the co-ordinates x and y are now used in a different sense compared with the earlier models). For this situation, the appropriate stress components are

$$p_{yy} = M\frac{\partial v}{\partial y} \quad (38)$$

and

$$p_{xy} = L\frac{\partial v}{\partial x} \quad (39)$$

where M and L are essentially tensile and shear moduli for this anisotropic-type deformation; Equations 38 and 39 reflect the linear character of the problem. The force equilibrium equation is

$$\frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{xy}}{\partial x} = 0 \quad (40)$$

which, after substituting from Equations 38 and 39, becomes

$$M\frac{\partial^2 v}{\partial y^2} + L\frac{\partial^2 v}{\partial x^2} = 0 \quad (41)$$

an equation that is analogous to the differential equations which govern the behaviour of the discrete element models analysed earlier in this section. Upon introducing the new variable $X = x(M/L)^{1/2}$, this equation reduced to Laplace's equation in the variables X and y

$$\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial X^2} = 0 \quad (42)$$

Against this background, consider the particular model (Fig. 4) of the plane strain deformation of a slab of thickness $2h$ in the y direction and of infinite dimensions in the x and z directions. The slab contains a crack, with infinite length in the z direction, and of length $2c$ in the x direction. The slab surfaces are subject to the displacements $\pm D_*$, such that the macroscopic strain applied to the slab is $\varepsilon_M = D_*/h$. By appealing to the results [4] for the Mode III deformation of this slab, when the displacement is now parallel to the z-axis, a situation for which Laplace's equation is governing, it follows that the strain $\varepsilon_L = \partial v/\partial y$ along the plane of the crack at a small distance b ahead of the crack tip is

$$\varepsilon_L = \frac{\partial v}{\partial y} = \frac{K}{M(2\pi b_*)^{1/2}} \quad (43)$$

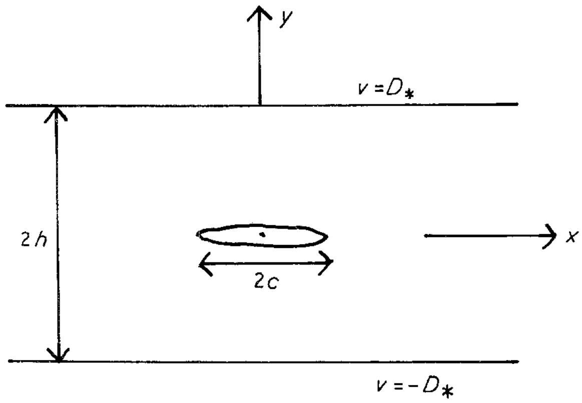

Figure 4 The continuum model of a crack in a slab with finite thickness $2h$. The elements whose behaviour is averaged are parallel to the y-axis, and displacements are applied to the slab faces.

with $b_* = b(M/L)^{1/2}$ and where K is given by the expression

$$K = M\varepsilon_M \left(2h \tanh \frac{\pi C_*}{2h}\right)^{1/2} \quad (44)$$

with $C_* = C(M/L)^{1/2}$. Equations 43 and 44 then give

$$\frac{\varepsilon_L}{\varepsilon_M} = \left[h \tanh \frac{\pi C}{2h}\left(\frac{M}{L}\right)^{1/2} \Big/ \pi b \left(\frac{M}{L}\right)^{1/2}\right]^{1/2} \quad (45)$$

This result shows that $\varepsilon_L/\varepsilon_M$ decreases as the intercommunication (L) becomes weaker, but that $\varepsilon_L/\varepsilon_M$ is relatively insensitive to the strength of the shear intercommunication for a wide range of h values, having a magnitude relevant to the limiting case where the elements are of infinite length, i.e. $h \to \infty$, when Equation 45 reduces to

$$\frac{\varepsilon_L}{\varepsilon_M} = \left(\frac{C}{2b}\right)^{1/2} \quad (46)$$

This result is compatible with those for the discrete element models analysed earlier in this section.

Now consider the case (Fig. 5) where, instead of displacements being applied to the slab faces, normal stresses, σ, are applied, these generating a far-field tensile strain of magnitude $\varepsilon_A = \sigma/M$. In this case, proceeding as for the model in Fig. 4, the parameter K is given by the expression [5]

$$K = M\varepsilon_A \left(2h \tanh \frac{\pi C_*}{2h}\right)^{1/2} K\left[\tanh \frac{\pi C_*}{2h}\right] \Big/ \left[\frac{\pi}{2}\right] \quad (47)$$

where, K, on the right-hand side of Equation 47, is the appropriate complete elliptic integral. The result analogous to Equation 45 is

$$\frac{\varepsilon_L}{\varepsilon_M} = \left[h \tanh \frac{\pi C}{2h}\left(\frac{M}{L}\right)^{1/2} \Big/ \pi a \left(\frac{M}{L}\right)^{1/2}\right]^{1/2} \times K\left[\tanh \frac{\pi C}{2h}\left(\frac{M}{L}\right)^{1/2}\right] \Big/ \left[\frac{\pi}{2}\right] \quad (48)$$

This result again demonstrates that $\varepsilon_L/\varepsilon_M$ decreases as the intercommunication becomes weaker, but that $\varepsilon_L/\varepsilon_M$ is relatively insensitive to the strength of the shear intercommunication for a wide range of h values, having a magnitude appropriate to the limiting case where the elements have infinite length, i.e. $h \to \infty$, when Equation 48 reduces to

$$\frac{\varepsilon_L}{\varepsilon_A} = \left(\frac{C}{2b}\right)^{1/2} \quad (49)$$

Furthermore, comparison of Equations 46 and 49 shows that the results for the macroscopic and far-field strain situations are identical. These results are also compatible with the conclusions from the analyses for the discrete element models.

3. Discussion

The preceding section's analyses, both for the discrete element models and models where the behaviour of the discrete elements is averaged, have been concerned with the case where the shear intercommunication between the elements is linear, in the sense that the shear resistance is linearly related to the shear strain. The effect of the strength of the shear intercommunication on a material's resistance to crack extension has been investigated, and it has been shown that the crack extension resistance, increases as the intercommunication becomes weaker, though there is a wide range of situations where this resistance is relatively insensitive to the strength of the shear intercommunication.

This means that when seeking explanations for large differences in the crack extension resistances of various types of material, it is necessary to move beyond the confines of a simple linear intercommunication law, and view such a comparison in terms of different types of force law. Thus, as emphasized by Gordon [1], there are essentially three categories of material that are highly resistant to crack extension. Firstly, where the intercommunication is very weak to the extent that the structural elements are essentially isolated; examples are rope, cloth, and natural tendons. Secondly, where a structure contains weak interfaces so that the adhesion between the elements fails fairly readily; examples are timber, teeth and some artificial composite materials. Thirdly, where the intercommunication is weak at low stresses followed by a rapid increase at a critical strain level; examples

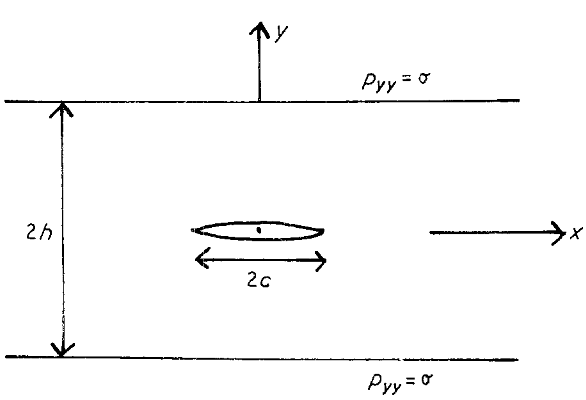

Figure 5 The continuum model of a crack in a slab with finite thickness $2h$. The elements whose behaviour is averaged are parallel to the y-axis, and stresses are applied to the slab faces.

are many human and animal membranes. These three extreme categories emphasize the necessity of proceeding beyond a simple linear intercommunication force law, when considering the effects of intercommunication on crack extension resistance.

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